Motivic invariants for moduli of parabolic Higgs bundles and moduli of parabolic connections on a curve.

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Introduction

Based on joint work with Roman Fedorov and Yan Soibelman.

We are interested in computing certain invariants of moduli spaces (or rather moduli stacks) of Higgs bundles and vector bundles with connections on a curve.

Sources of Motivation:

- Motivic Donaldson-Thomas invariants. These were developed by Kontsevich and Soibelman for 3-dimensional Calabi-Yau categories and also Joyce and Song for abelian categories of coherent sheaves on compact Calabi-Yau 3-folds. They occur in physics as "refined BPS invariants".
- The work of Hausel, Letellier, Rodriguez-Villegas, and others regarding mixed Hodge polynomials of character varieties.
- Point counting for algebraic varieties and stacks over a finite field $\mathbb{F}_q$. Specifically, computations done for moduli stacks of Higgs bundles done by Mozgovoy, Schiffmann, and Mellit.

Counting points of varieties over a finite field

One way of looking at motivic class computations is as a generalization of counting the number of points in an algebraic variety over a finite field.

Example

Let $|X| := |X(\mathbb{F}_q)|$ denote the number of rational points of an algebraic variety X over a finite field $\mathbb{F}_q$ with q elements.

- $|\mathbb{A}^n| = q^n$.
- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\operatorname{GL}(n)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$ considered as the space of n linearly independent columns.
- $|\mathrm{Gr}(d, n)| = \frac{[n!]_q}{[d!]_q[(n-d)!]_q}$, where $[n!]_q = \prod_{i=1}^n \frac{q^i - 1}{q-1}$, by presenting $\mathrm{Gr}(d, n)$ a single orbit under a $\operatorname{GL}(n)$ action.

Counting points of algebraic stacks over a finite field

- $|\mathbb{P}^n|$ can also be computed by presenting $\mathbb{P}^n$ as a quotient of $\mathbb{A}^n - \{0\}$ by the action of $\mathrm{GL}(1)$.
- We can try to do similar computations even when the group action is "bad" and the quotient is no longer a variety.
- Algebraic stacks generalize varieties by allowing points to have nontrivial automorphisms.
- Examples include quotients X/G of a variety by a non-free algebraic group action, moduli of curves, moduli of vector bundles, etc.

If $\mathcal{Y}$ is an algebraic stack, we can try generalize point counting for varieties by defining the **volume** (also known as the **mass**) of $\mathcal{Y}$ as a weighted sum over all its $\mathbb{F}_q$-rational points:

$$|\mathcal{Y}| := \sum_{y \in \mathcal{Y}} \frac{1}{|\operatorname{Aut}(y)|}.$$

This is not guaranteed to converge. However, all the stacks we consider are of finite type, for which it converges.

Example: Nilpotent endomorphisms

Let $\mathcal{N}_n$ be the variety consisting of $n \times n$ nilpotent matrices. The group $\mathrm{GL}(n)$ acts on $\mathcal{N}_n$ by conjugation.

- Each partition λ of n defines a unique Jordan normal form $N_\lambda \in \mathcal{N}_n$.
- Let $\mathcal{N}_\lambda$ be the conjugacy class of N_λ, and let $\mathrm{Aut}(\lambda)$ be its stabilizer.
- Since $|\mathcal{N}_\lambda| = \frac{|\mathrm{GL}(n)|}{|\mathrm{Aut}(\lambda)|}$, the volume of the quotient stack is:

$$|\mathcal{N}_n/\mathrm{GL}(n)| = \sum_{\substack{\lambda \\ |\lambda|=n}} \frac{|\mathcal{N}_\lambda|}{|\mathrm{GL}(n)|} = \frac{|\mathcal{N}_n|}{|\mathrm{GL}(n)|} = \frac{q^{\frac{n(n-1)}{2}}}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}.$$

- If we instead consider the generating function $N(t) = \sum_{n\geq 0} |\mathcal{N}_n/GL(n)| t^n$, then we get:

$$N(t) = .\sum_{n\geq 0} \frac{1}{(q^n-1)(q^{n-1}-1)\cdots(q-1)} t^n = \mathrm{Exp}\left(\frac{t}{q-1}\right),$$

where Exp is the plethystic exponential. Note that Exp has an inverse function Log, known as the plethystic logarithm.

Example: Volume of the stack of vector bundles

A more complicated example involves the stack of vector bundles on a curve.

Siegel formula

Let X be a smooth projective curve of genus g. Let $\mathrm{Bun}_{r,d}(X)$ be the moduli stack parametrizing isomorphism classes of rank r, degree d vector bundles over X. The volume over $\mathbb{F}_q$ may be computed as:

$$|\mathrm{Bun}_{r,d}(X)| = \frac{q^{(r^2-1)(g-1)}}{q-1} |\mathrm{Jac}(X)| \prod_{i=2}^{r} Z_X(q^{-i}),$$

where $Z_X(z)$ is the Hasse-Weil zeta function defined as:

$$Z_X(t) := \exp\left(\sum_{m \geq 1} |X(\mathbb{F}_{q^m})| t^m\right) = \sum_{n=0}^{\infty} |X^{(n)}| t^n,$$

while $\mathrm{Jac}(X)$ is the Jacobian of X and $X^{(n)}$ is the n-th symmetric power of X.

Grothendieck ring of varieties

We can consider the following:

Grothendieck Ring

Let $K_0(Var_k)$ be the abelian group generated by isomorphism classes $[X]$ of varieties over k, subject to the scissor relations

$$[X] = [X \backslash Z] + [Z],$$

for any closed subvariety $Z \subset X$. This group carries a natural commutative ring structure defined by

$$[X] \cdot [Y] = [X \times_k Y],$$

with the unit element $[\mathrm{Spec}\, k] = 1$. The element $[X] \in K_0(Var_k)$ is called the **motivic class** of X.

The Grothendieck ring $K_0(Var_k)$ has many interesting properties. For example, a result of Bittner says that for $\mathrm{char}(k) = 0$, it is generated by classes of smooth, connected, projective varieties.

Realizations of the Grothendieck ring

Invariants with values in a commutative ring A that respect the Grothendieck ring operations gives rise to realization morphisms $K_0(Var_k) \to A$.

Example

- Point counting: For $k = \mathbb{F}_q$ define $\# : K_0(Var_k) \to \mathbb{Z}$ as $\#([X]) = |X(\mathbb{F}_q)|$, the number of rational points over $\mathbb{F}_q$.
- Euler characteristic: For $k = \mathbb{C}$ define $\chi : K_0(Var_k) \to \mathbb{Z}$ such that $\chi([X]) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q})$ is the Euler characteristic of X.
- Hodge-Deligne polynomial: For $k = \mathbb{C}$, define $E : K_0(Var_k) \to \mathbb{Z}[u, v]$ by setting $E([X]) = \sum_{p,q} (-1)^{p+q} h^{p,q} u^p v^q$ such that $h^{p,q} = \dim H^q(X, \wedge^p \Omega^1_X)$ for any smooth projective variety X.

Motivic Classes of Varieties

So, instead of performing computations of invariants directly, we can instead perform computations of motivic classes in $K_0(Var_k)$. Letting $[\mathbb{A}^1] = \mathbb{L}$, we see that the same methods used for counting points over $\mathbb{F}_q$ give us:

Example

- $[\mathbb{A}^n] = \mathbb{L}^n$
- $[\mathbb{P}^n] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots \mathbb{L}^n$
- $[\mathrm{GL}_n(k)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$
- $[\mathrm{Gr}(d,n)] = \frac{[n!]_\mathbb{L}}{[d!]_\mathbb{L}[(n-d)!]_\mathbb{L}}$, where $[n!]_\mathbb{L} = \prod_{i=1}^n \frac{\mathbb{L}^i - 1}{\mathbb{L} - 1}$

We are interested in performing similar computations for algebraic stacks. However, we run into problems because points have automorphisms.

Motivic classes of stacks

We want to compute a motivic analogue of the volume of an algebraic stack. We can do this in some specific cases as follows:

- For a quotient stack $\mathcal{Y} = Y/\operatorname{GL}(n)$ resulting from the action of $\operatorname{GL}(n)$ on an algebraic variety Y, we define $[\mathcal{Y}] = \frac{[Y]}{[\operatorname{GL}(n)]}$.
- This does not exist in $K_0(Var_k)$, so we pass to a certain completion $\hat{K}_0(Var_k)$ of $K_0(Var_k)[\frac{1}{\mathbb{L}}]$, in which $\frac{1}{\mathbb{L}}$ is a small parameter and where $[\operatorname{GL}(n)]$ has an inverse.
- In general, given a nice enough algebraic stack $\mathcal{Y}$, we can stratify it as $\mathcal{Y} = \sqcup_{i=1}^{\infty} Y_i/\operatorname{GL}(n_i)$ and define

$$[\mathcal{Y}] = \sum_{i=1}^{\infty} [Y_i]/[\operatorname{GL}(n_i)].$$

- It can be shown (for nice enough stacks) this converges and is independent of stratification. Specifically, this covers all the examples we are interested in.

Motivic class of the stack of vector bundles

Let X be a smooth projective curve of genus g as before. There is a motivic version of the Siegel formula for the volume of the stack of vector bundles.

Theorem (Behrend-Dhillon, 2007)

We have:

$$[Bun_{r,d}(X)] = \mathbb{L}^{(r^2-1)(g-1)} \frac{[Jac(X)]}{\mathbb{L}-1} \prod_{i=2}^{r} Z_X(\mathbb{L}^{-i}),$$

where $Z_X(z)$ is Kapranov's motivic zeta function defined as:

$$Z_X(z) = \sum_{n=0}^{\infty} [X^{(n)}] z^n,$$

while $Jac(X)$ is the Jacobian of X and $X^{(n)}$ is the n-th symmetric power of X as before.

Higgs Bundles

Let X be a smooth projective curve of genus g over a field k as before. We are interested in motivic class computations for objects consisting of vector bundles with additional data.

- A Higgs bundle on X is a pair (E, Φ) such that E is a vector bundle and $\Phi : E \to E \otimes \Omega^1_X$ is an $\mathcal{O}_X$-linear morphism (a Higgs field).
- Let D be divisor on X. A D-twisted Higgs bundle is a (E, Φ) such that E is a vector bundle and $\Phi : E \to E \otimes \Omega^1_X(D)$ is an $\mathcal{O}_X$-linear morphism (a D-twisted Higgs field).
- The notion of slope stability for vector bundles extends to Higgs bundles. That is, any proper subbundle $F \subset E$ preserved by Φ must satisfy the inequality $\frac{\deg F}{\operatorname{rk} F} < \frac{\deg E}{\operatorname{rk} E}$ to be considered stable (non-strict inequality for semistablity).

Denote by $\mathcal{M}^{ss}(X, D, r, d)$ the moduli space of semistable D-twisted Higgs bundles such that $\operatorname{rk} E = r$ and $\deg E = d$. The following formula was conjectured by Mozgovoy.

Mozgovoy Conjecture

(Mozgovoy 2011)

Let $\deg D = 2g - 2 + p$ for $p \geq 0$ and let r, d be coprime. Let $\mathcal{H}_\lambda^{(p)}(t)$ be defined as

$$\mathcal{H}_\lambda^{(p)}(t) = (-1)^{p|\lambda|} t^{(1-g)(2n(\lambda)+|\lambda|)+p(n(\lambda')-n(\lambda))} \mathbb{L}^{pn(\lambda')} \prod_{s \in d(\lambda)} Z_X(t^{h(s)} \mathbb{L}^{a(s)}),$$

where $d(\lambda)$ is the Young diagram of λ, $a(s) = \lambda_i - j$, $l(s) = \lambda'_j - i$, $h(s) = a(s) + l(s) + 1$, and $n(\lambda) = \sum_{s \in d(\lambda)} l(s)$. Define $H_r^{(p)}(t)$ via the generating function

$$\operatorname{Exp}\left(\sum_{r \geq 1} \frac{(-1)^{pr} t^{(1-g)r^2 - p\binom{r}{2}} H_r(t)}{(1-t)(1-t\mathbb{L})} T^r \right) = \sum_\lambda \mathcal{H}_\lambda^{(p)}(t) T^{|\lambda|},$$

where Exp is the plethystic exponential. We have $H_r^{(p)}(t)$ are polynomials and

$$[\mathcal{M}^{ss}(X, D, r, d)] = \mathbb{L}^{(g-1)r^2 + p\binom{r}{2} + 1} H_r^{(p)}(1).$$

Further computations for Higgs Bundles

There have been related computations done for the number of points over a finite field for moduli spaces and moduli stacks of Higgs bundles on a curve.

- (Mozgovoy, Schiffmann 2014) - different formula for volume of moduli stack of semistable twisted Higgs bundles over a finite field.
- (Mellit 2017) - showed Mozgovoy's formula coincides with Mozgovoy and Schiffmann's formula in the finite field case.
- (Mellit 2017) - computed the volume of the moduli stack of semistable parabolic Higgs bundles over a finite field.
- (Mozgovoy, O'Gorman 2019) - proved Mozgovoy's conjecture for twisted Higgs bundles in the finite field case.

Our results concern motivic versions of the Mozgovoy and Schiffmann computation for Higgs bundles and the Mellit computations for parabolic Higgs bundles.

Motivic classes for moduli stacks of Higgs Bundles and connections

From now on assume k has characteristic 0. Consider the following:

- Let E be a vector bundle on X. We define a connection on X as a k-linear morphism $\nabla : E \to E \otimes \Omega^1_X$ satisfying the Leibnitz rule $\nabla(fs) = df \otimes s + f \otimes \nabla(s)$ for $s \in E$ and $f \in \mathcal{O}_X$.
- $\mathrm{Conn}_r(X)$ - moduli stack of rank r vector bundles with connections on X. That is, $\mathrm{Conn}_r(X)$ parametrizes up to isomorphism pairs (E, ∇), where E is a rank r vector bundle on X and ∇ is a connection on E.
- $\mathcal{M}^{ss}_{r,d}(X)$ - moduli stack of semistable Higgs bundles of rank r and degree d on X. That is, $\mathcal{M}^{ss}_{r,d}(X)$ parametrizes up to isomorphism pairs (E, Φ), where $\mathrm{rk}\ E = r$, $\deg\ E = d$, and $\Phi : E \to E \otimes \Omega^1_X$ is a Higgs field satisfying the slope semistability condition.

The motivic classes for these moduli stacks are related as follows:

Theorem (Fedorov-Soibelman-S, 2017)

$$[Conn_r(X)] = [\mathcal{M}^{ss}_{r,0}(X)]$$

Formula for Semistable Higgs Bundles

Theorem 1 (Fedorov-Soibelman-S, 2017)

For $\mathrm{char}(k) = 0$ and sufficiently large e, we have $[\mathcal{M}^{ss}_{r,d}(X)] = H_{r,d+er}$, where $H_{r,d}$ is defined via the generating function

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H_{r,d} w^r z^d = \operatorname{Exp}\left(\sum_{d/r=\tau} B_{r,d} w^r z^d \right),$$

where Exp *is the plethystic exponential and τ is a rational number. Similarly $B_{r,d}$ is given by*

$$\sum_{\substack{r,d\in\mathbb{Z}_{\geq 0} \\ (r,d)\neq(0,0)}} B_{r,d} w^r z^d = \mathbb{L} \operatorname{Log}\left(\sum_{\lambda} \mathbb{L}^{(g-1)\langle\lambda,\lambda\rangle} J^{mot}_{\lambda}(z) H^{mot}_{\lambda}(z) w^{|\lambda|} \right).$$

Here $J^{mot}_{\lambda}(z), H^{mot}_{\lambda}(z)$ are rational functions written in terms of Z_X and Log *is the plethystic logarithm. The sum is over all partitions and $\langle\lambda,\lambda\rangle = \sum_i (\lambda'_i)^2$.*

Parabolic vector bundles

Consider the following generalization of vector bundles on a curve:

- Let X be as before and let $D = x_1 + \cdots + x_n$ be a reduced divisor on X. A **parabolic bundle** consists of a pair $\mathbf{E} = (E, E_{i,j})$ such that E is a vector bundle on X and a filtration of the fiber $E_{x_i} = E_{i,0} \supset \cdots \supset 0$ by subspaces.
- Denote by $\gamma = (\gamma_{ij})$ the dimension vector of $\mathbf{E}$, where $\gamma_{i0} = \gamma_0 = \operatorname{rk} E$ and $\gamma_{ij} = \dim E_{i,j}$.
- Let $\sigma = (\sigma_{ij})$ be a real vector, $\varkappa$ a nonnegative real number. Define the $(\varkappa, \sigma)$-degree of $\mathbf{E}$ as $\deg_{\varkappa,\sigma} \mathbf{E} = \varkappa d + \sum_{i,j} \sigma_{ij}(\gamma_{ij} - \gamma_{ij+1})$.
- We can extend the notion of slope stability to parabolic bundles by defining the slope of $\mathbf{E}$ with respect to stability parameter σ as $\mu_\sigma(\mathbf{E}) = \frac{\deg_{1,\sigma} \mathbf{E}}{\operatorname{rk} E}$.

Parabolic connections and parabolic Higgs bundles

Let X and k be as before. Given a parabolic bundle $\mathbf{E} = (E, E_{ij})$ on X, one can define Higgs bundles and connections compatible with the parabolic structure. Let $D = x_1 + \cdots + x_n$ be the corresponding divisor and let $\zeta := (\zeta_{ij})$ be a vector with entries in k.

- A ζ-**parabolic connection** on $\mathbf{E}$ is a k-linear morphism $\nabla : E \to E \otimes \Omega^1_X(D)$ satisfying the Leibnitz rule such that

$$(\mathrm{Res}_{x_i} \nabla - \zeta_{ij} \cdot \mathrm{Id}) E_{i,j} \subset E_{i,j+1},$$

  here $\mathrm{Res}_{x_i} \nabla \in \mathrm{End}(E_{x_i})$ is the residue of ∇ at x_i.

- A ζ-**parabolic Higgs bundle** is a pair $(\mathbf{E}, \Phi)$ such that $\Phi : E \to E \otimes \Omega^1_X(D)$ is twisted Higgs field satisfying

$$(\mathrm{Res}_{x_i} \Phi - \zeta_{ij} \cdot \mathrm{Id}) E_{i,j} \subset E_{i,j+1},$$

  where $\mathrm{Res}_{x_i} \Phi \in \mathrm{End}(E_{x_i})$ is the residue of Φ at x_i.

Moduli stacks of parabolic connections and parabolic Higgs bundles

We are interested in the motivic classes of the following stacks:

- $\mathrm{Conn}_{\gamma,\zeta}(X,D)$ - moduli stack parametrizing isomorphism classes of pairs $(\mathbf{E},\nabla)$ where $\mathbf{E}$ is a parabolic bundle with dimension vector γ and ∇ is a ζ-parabolic connection on $\mathbf{E}$.
- $\mathcal{M}^{ss}_{\gamma,d,\zeta,\sigma}(X,D)$ - moduli stack parametrizing isomorphism classes of ζ-parabolic Higgs bundles (E,Φ) that are semistable with respect to stability parameter σ.

As before, in characteristic 0 we have the following theorem:

Theorem (Fedorov-Soibelman-S, 2019)

Assume k is not a finite extension of $\mathbb{Q}$. Given a vector $\zeta=(\zeta_{ij})$ with entries in k, there another vector $\zeta'=(\zeta'_{ij})$ such that

$$[Conn_{\gamma,\zeta'}(X,D)]=[\mathcal{M}^{ss}_{\gamma,d,\zeta,\sigma}(X,D)].$$

Motivic classes for parabolic Higgs bundles

Theorem 2 (Fedorov-Soibelman-S, 2019)

For $char(k) = 0$ *and* e *large enough, we have* $[\mathcal{M}^{ss}_{\gamma,d,\zeta,\sigma}(X, D)] = H_{\gamma,d-e\gamma_0}(\zeta, \sigma)$*, where* $H_{\gamma,d}(\zeta, \sigma)$ *is defined via the generating function*

$$\sum_{\substack{d\le 0,\gamma \\ \deg_{0,\zeta}\gamma=0 \\ \deg_{1,\sigma}\gamma=\tau\gamma_0}} \mathbb{L}^{-\chi(\gamma)} H_{\gamma,d}(\zeta,\sigma) w^{\gamma} z^{d} = \operatorname{Exp}\left(\sum_{\substack{d\le 0,\gamma \\ \deg_{0,\zeta}\gamma=0 \\ \deg_{1,\sigma}\gamma=\tau\gamma_0}} \overline{B}_{\gamma,d} w^{\gamma} z^{d} \right).$$

where sums are taken over all dimension vectors γ, $w^{\gamma} = w_0^{\gamma_0}\prod_i\prod_j w_{i,j}^{\gamma_{ij}-\gamma_{ij+1}}$*, and* $\chi(\gamma) := (g-1)\gamma_0^2 + \sum_i\sum_{j<j'}\gamma_{ij}\gamma_{ij'}$*. Similarly* $\overline{B}_{\gamma,d}$ *is given by*

$$\sum_{d\le 0,\gamma} \overline{B}_{\gamma,d} w^{\gamma} z^{d} = \mathbb{L}\cdot \operatorname{Log}\left(\sum_{\lambda} w^{|\lambda|} J^{mot}_{\lambda}(z^{-1}) H^{mot}_{\lambda}(z^{-1}) \prod_{x_i} \tilde{H}^{mot}_{\lambda}(w_{i,j}; z^{-1}) \right),$$

where $\tilde{H}^{mot}_{\lambda}(w_{i,j}; z^{-1})$ *are motivic modified Macdonald polynomials.*

The Deligne-Simpson problem

Consider the following question posed in a 1989 letter from Deligne to Simpson:

(multiplicative) Deligne-Simpson problem

Given conjugacy classes $C_1, \cdots, C_n$ of invertible matrices in $\mathrm{GL}(r, \mathbb{C})$, do there exist representatives $A_1 \in C_1, \ldots, A_n \in C_n$ such that

$$A_1 \cdot A_2 \cdots A_n = \mathrm{Id}?$$

This question may be reinterpreted as

Given a reduced divisor $D = x_1 + \cdots + x_n$ on $\mathbb{P}^1$ and conjugacy classes $C_1, \cdots, C_n$ of $r \times r$ matrices, does there exist a rank r vector bundle E with logarithmic connection $\nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ such that $\mathrm{Res}_{x_i} \nabla \in C_i$.

by looking at the monodromy of ∇. If the conjugacy classes C_i are all semisimple, then such ∇ are parabolic connections, so the Deligne-Simpson problem is equivalent to asking when $\mathrm{Conn}_{\gamma,\zeta}(X, D)$ has points over $\mathbb{C}$. This happens precisely when $[\mathrm{Conn}_{\gamma,\zeta}(X, D)] \neq 0$.

Further Directions: The irregular parabolic case

Let X and k be as before. One can generalize the definition of parabolic vector bundles, parabolic Higgs bundles, and parabolic connections to the case when the divisor D is no longer reduced.

- Let $D = \sum_{i=1}^{l} n_i \cdot x_i$, where $n_i \geq 1$, be a divisor on X. An **irregular parabolic bundle** $\mathbf{E} = (E, E_{i,j})$ on X consists of a vector bundle E and a filtration
$$E|_{n_i \cdot x_i} = E_{i,0} \supset E_{i,1} \supset \cdots \supset 0$$
by locally free $\mathcal{O}_{n_i \cdot x_i}$-modules.
- Let $\zeta = (\zeta_{ij})$ be a vector consisting of entries $\zeta_{ij} \in H^0(n_i \cdot x_i, \Omega^1_X(D)|_{n_i \cdot x_i})$.
- Let $\mathbf{E}$ be an irregular parabolic bundle and let $\Phi : E \to E \otimes \Omega^1_X(D)$ be a twisted Higgs field. We say $(\mathbf{E}, \Phi)$ is an **irregular ζ-parabolic Higgs bundle** if
$$(\Phi|_{n_i \cdot x_i} - 1 \otimes \zeta_{i,j})(E_{i,j}) \subset E_{i,j+1} \otimes \Omega^1_X(D)|_{n_i \cdot x_i}.$$
- One can give a similar definition for an **irregular ζ-parabolic connection**.
- Can also define stability using a parameter $\sigma = (\sigma_{ij})$ as before.

Further Directions: The irregular parabolic case (continued)

Questions:

- What is the relationship between the motivic classes of the moduli stacks of irregular parabolic bundles with irregular parabolic connections and semistable irregular parabolic Higgs bundles? That is, describe a nice enough necessary and sufficient condition for irregular ζ-parabolic Higgs fields/connections to exist.
- Give an exact formula for the motivic class of the moduli stack of semistable irregular parabolic Higgs bundles. What is the analogue of the modified Macdonald polynomials in this case?
- What does this say about the irregular Deligne-Simpson problem? That is, what is the relationship between irregular singular connections on $\mathbb{P}^1$ with fixed normal forms for local data and irregular parabolic connections?

Thank You!